The Lee Fields Medal V: SOLUTIONS

1. Find three integers in arithmetic sequence whose product is a prime.

   Solution: It seems this is impossible. Let n_1, n_2, n_3 be the integers. How could $n_1 n_2 n_3$ be prime? Well, two of them will have to be something like one, so maybe $n_1 = 1$ and $n_2 = -1$. It then follows that $n_3 = -3$ and this works:

   $$n_1 n_2 n_3 = (1)(-1)(-3) = +3,$$

   which is prime.

2. Consider a unit circle on the plane of radius $r = 1$ and centre $O(0,0)$ and a rotation angle $\theta \in [0, 2\pi]$. Starting with the point $x_0 = (1,0)$, rotate x_0 through the angle θ to get a new point x_1. Then rotate x_1 through θ to get a new point x_2. Then rotate x_2 through θ to get x_3, etc. For example, for $\theta = \pi/2 = 90°$:

   $$\begin{aligned} x_0 &= (1,0) \\ x_1 &= (0,1) \\ x_2 &= (-1,0) \\ x_3 &= (0,-1) \\ x_4 &= (1,0), \end{aligned}$$

   and the sequence of points repeats itself in this case.

   Show that no matter what rotation angle θ you pick, for large enough N, the sequence

   $$x_0,\ x_1,\ x_2,\ x_3, \ldots, x_N$$

   contains two points such that the distance between the two points is less than $1/100$.

   Solution: Let us show that $N = 630$ suffices to force two points within a distance of $1/100$.

   The circumference of the circle is $L = 2\pi r = 2\pi$. Cut the circumference into the following 'bins':

   $$[0, 2\pi) = [0, 1/100) \sqcup [1/100, 2/100) \sqcup [2/100, 3/100) \sqcup \cdots \sqcup [6.27 - 6.28) \sqcup [6.28, 2\pi).$$

Note 6.28 is the largest multiple of 0.01 smaller than 2π. There are 629 bins here. Let $N = 629$ and consider the 630 points:

$$x_0, x_1, x_2, \ldots, x_{630}.$$

There are 630 points, but only 629 bins: so at least one bin must contain two or more points (this is the *pigeonhole principle*), say x_i and x_j. Therefore the arc-distance is:

$$d_{\text{arc}}(x_i, x_j) < 1/100.$$

Now, a circle is *convex* meaning that the straight-line distance is less than the arc-distance:

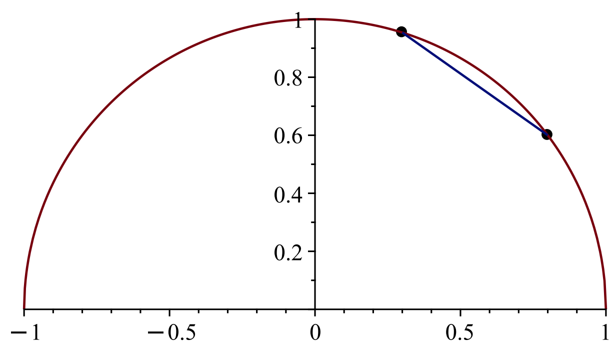


Figure 1: The straight-line distance between x_i and x_j is less than the arc-distance.

Therefore:

$$d(x_i, x_j) < d_{\text{arc}}(x_i, x_j) < 1/100.$$

3. Suppose that i is a number such that $i^2 = -1$. Suppose further that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ and}$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Show that

$$e^{it} = \cos(t) + i\,\sin(t),$$

and that

$$e^{i\pi} + 1 = 0.$$

Solution: Note first that

$$i^3 = i^2 \cdot i = -1 \cdot i = -i, \text{ and}$$
$$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = +1,$$

and the sequence repeats after this, $i^5 = i$, $i^6 = i^2 = -1$, $i^7 = i^3 = -i$, etc. Now substitute $x = it$ into the infinite series for e^x:

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\ &= 1 + it + \frac{-t^2}{2!} + \frac{-it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) \\ &= \cos(t) + i\sin(t) \end{aligned}$$

Now substitute $t = \pi$ recalling $\cos(\pi) = -1$ and $\sin(\pi) = 0$:

$$\begin{aligned} e^{i\pi} &= \cos(\pi) + i\sin(\pi) \\ \implies e^{i\pi} &= -1 + i(0) \\ \implies e^{i\pi} + 1 &= 0 \end{aligned}$$

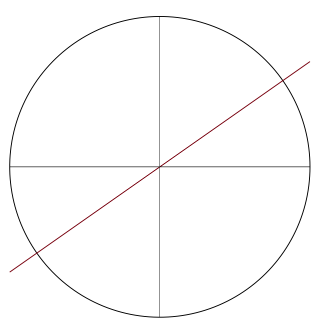
4. Let r, $m > 0$ be real numbers

   (a) Produce a rough sketch of the curves $y = mx$ and $x^2 + y^2 = r^2$.

   (b) Using this rough sketch, write down how many solutions the below set of simultaneous equations has:

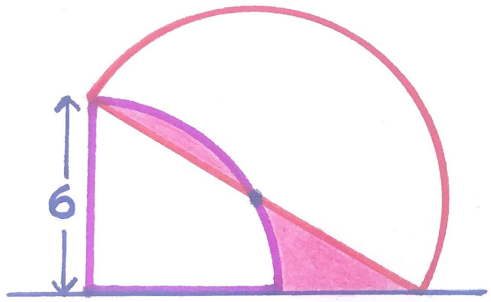
$$\begin{aligned} y &= mx \\ x^2 + y^2 &= r^2 \end{aligned}$$

Solution: Here we have a straight-line through the origin, of slope $m > 0$, and a circle of centre $O(0, 0)$ and radius r:



Points (x_0, y_0) on curves corresponds to solutions of equations of curves. Points on both curves — intersections — satisfy both equations. Therefore the number of solutions is the same as the number of intersections. In this case, two.

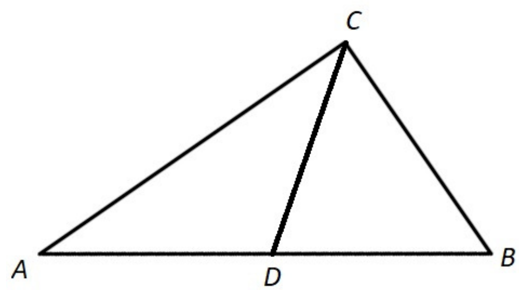
5. The centre of the semicircle lies on the perimeter of the quarter circle. What's the shaded area?



Solution: First we will prove the following.

> *If $[AB]$ is a diameter of a circle, and ΔACB is right-angled at C, then C is on the circle.*

Proof. Let ACB be a right-angled triangle, and D the midpoint of $[AB]$ (in other words the centre of the circle):



Note that if $[AB]$ is a diameter, then $|AB| = 2r$, and $|AD| = |BD| = r$. By Pythagoras:

$$|AB|^2 = |AC|^2 + |BC|^2 \implies |AC|^2 + |BC|^2 = 4r^2 \qquad (\star).$$

On the other hand, by two cosine rules:

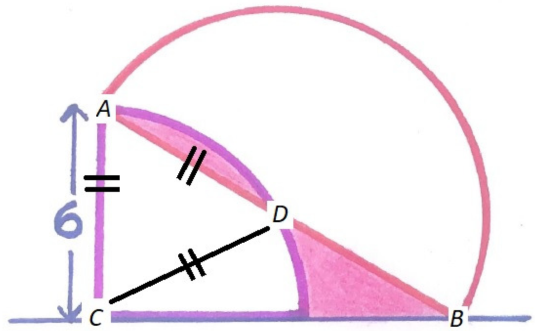
$$\begin{aligned}
&|AC|^2 = |AD|^2 + |CD|^2 - 2|AD||CD|\cos(\angle ADC) \\
\implies &|AC|^2 = r^2 + |CD|^2 - 2r|CD|\cos(\angle ADC) \\
&|BC|^2 = |BD|^2 + |CD|^2 - 2|BD||CD|\cos(\angle BDC) \\
\implies &|BC|^2 = r^2 + |CD|^2 - 2r|CD|\cos(180^\circ - \angle ADC) \\
\underset{\cos(180^\circ-\theta)=-\cos\theta}{\implies} &|BC|^2 = r^2 + |CD|^2 + 2r|CD|\cos(\angle ADC)
\end{aligned}$$

Sum these:

$$\begin{aligned}
|AC|^2 + |BC|^2 &= 2r^2 + 2|CD|^2 \\
\underset{\star}{\implies} 4r^2 &= 2r^2 + 2|CD|^2 \\
\implies |CD| &= r,
\end{aligned}$$

which implies C is on the circle. $\square$

Proceed to get an equilateral triangle:



Let $\mathcal{C}_1$ be the semi-circle, of radius r, and $\mathcal{C}_2$ be the quarter-circle of radius six. The theorem above gives $|AD| = |CD| = r$ and also $|AC| = |CD| = 6$ because they are both on the quarter-circle. Therefore in fact

$$r = |AD| = |CD| = |AC| = 6.$$

That little shape inside $\mathcal{C}_2$ above $[AD]$ is a *lune*. This can be calculated using the area of the sector minus the area of the equilateral triangle:

$$\text{Area(lune)} = \text{Area(sector)} - \text{Area}(\Delta ACD) = \frac{1}{6}\pi r^2 - \frac{1}{2}|AC||DC|\sin(60^\circ) = 6\pi - 9\sqrt{3}.$$

The other area, that little corner, is given by:

$$\begin{aligned}\text{Area(corner)} &= \text{Area}(\Delta ABC) - (\text{Area}(\mathcal{C}_1) - \text{Area(lune)}) \\ &= \frac{1}{2}(12)(6)\sin(60^\circ) - \left(\frac{1}{4}\pi(6)^2 - (6\pi - 9\sqrt{3})\right) = 9\sqrt{3} - 3\pi.\end{aligned}$$

Therefore the required area is

$$A = A(\text{lune}) + A(\text{corner}) = 6\pi - 9\sqrt{3} + 9\sqrt{3} - 3\pi = 3\pi.$$

6. Prove that for all $0^\circ < \theta < 90^\circ$,

$$\tan\theta > \sin\theta.$$

Solution: Recalling that dividing by a small number gives you a big number, and the hypotenuse's length, $h > 0$, is greater than the length of the adjacent $a > 0$ (to the angle θ). Where $o > 0$ is the length of the opposite side:

$$\begin{aligned} & a < h \\ \Longrightarrow\; & \frac{1}{a} > \frac{1}{h} \\ \underset{\times o}{\Longrightarrow}\; & \frac{o}{a} > \frac{o}{h} \\ \Longrightarrow\; & \tan\theta > \sin\theta.\end{aligned}$$

7. Construct a finite list of data/numbers with a unique mode such that

$$\text{median} < \text{mean} < \text{mode}\,.$$

Solution: Think about how many numbers $\{x_1, x_2, \ldots, x_n\}$ you need. One number does not give three different statistics, neither does two. Consider $n = 3$. Necessarily, for a unique mode, two must be equal, say $x_1 = x_2$. To ensure the mode is big, assume $x_3 < x_2$ so we have:

$$x_1 = x_2 \text{ and } x_1 > x_3.$$

This doesn't work though, because x_1, x_1, x_3 has median x_2. So we need four numbers, two of which are equal:

$$(x_1, x_1, x_3, x_4).$$

For median less than mean:

$$\begin{aligned} &\frac{x_1 + x_3}{2} < \frac{2x_1 + x_3 + x_4}{4} \\ &\implies x_3 < x_4, \end{aligned}$$

contradicting the decreasing hypothesis. So consider $n = 5$:

$$(x_1, x_1, x_3, x_4, x_5).$$

Now we want median less than mean:

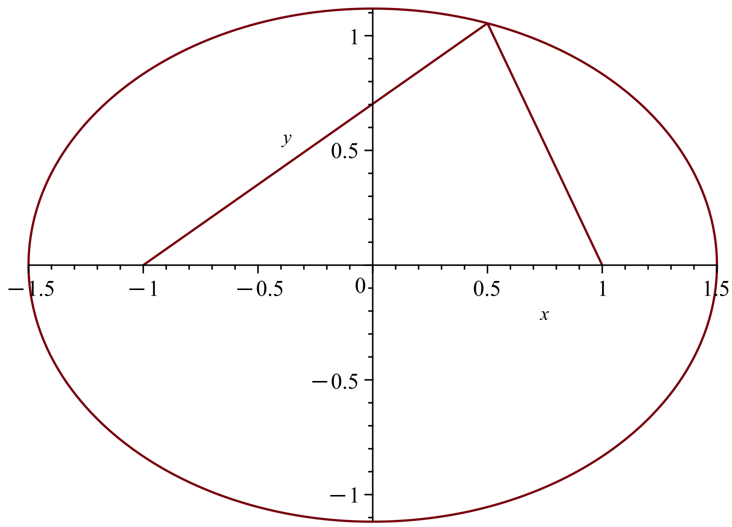
$$\begin{aligned} x_3 &< \frac{2x_1 + x_3 + x_4 + x_5}{5} \\ \underset{(\times 5/4)\circ(-x_3/5)}{\implies} x_3 &< \frac{2x_1 + x_4 + x_5}{4}, \end{aligned}$$

and we can achieve this by making x_1 large. Try $x_1 = 100$ and :

$$(100, 100, 2, 1, 0).$$

We have median two, mean $203/5 = 40.6$, and mode 100.

8. An ellipse is a curve surrounding two focal points F_1, and F_2, such that for all points P on the curve, the sum of the distances $|F_1P|$ and $|PF_2|$ is a constant. In the ellipse below, the foci are $(-1, 0)$ and $(1, 0)$, and the constant is four.



Write down the/an equation of the ellipse.

Solution: We want to write down that $P(x, y)$ is a point on the curve when:

$$|F_1P| + |F_2P| = 4.$$

But we know how to calculate distances!

$$\sqrt{(x-(-1))^2 + (y-0)^2} + \sqrt{(x-1)^2 + (y-0)^2} = 4$$
$$\implies \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4$$

9. A grandfather has one son and one daughter. The grandfather asks both his son and his daughter: "How many children do you have?". They answer, and he finds the mean-average of their answers. Then the grandfather asks all his grandchildren: "One of your parents is my child. How many children does that parent have?". They answer, and he finds the mean-average of their answers.

   (a) Show that the second answer is always greater than or equal to the first answer.
   (b) Under what conditions is the first answer equal to the second answer?

Solution: Let the son have m children and the daughter have n children. The answer to the first question is:

$$\frac{m+n}{2}.$$

For the second question, m children say m, and n children say n. So the answer is:

$$\frac{m\cdot m+n\cdot n}{m+n}=\frac{m^2+n^2}{m+n}.$$

We want to show that:

$$\frac{m^2+n^2}{m+n}>\frac{m+n}{2}.$$

Now, what we *can't* do is start with what we want to show, say P_n, and then end up with something true, say P_t, and then conclude that P_n is true. This is a fallacy. For example, P_n the statement $-1=1$, and you can get to P_t the statement $0=0$, but this doesn't mean P_n is true. We can start with something true, P_0, and then conclude, after a number of steps, that P_n is true. We can be guided *towards* P_0 by assuming P_n is true (but this is rough-work):

$$\begin{aligned}\frac{m^2+n^2}{m+n}&\geq\frac{m+n}{2}\\ \underset{\times(m+n)}{\Longrightarrow} m^2+n^2&\geq\frac{(m+n)^2}{2}\\ \underset{\times 2}{\Longrightarrow} 2(m^2+n^2)&\geq(m+n)^2\\ \Longrightarrow 2m^2+2n^2&\geq m^2+2mn+n^2\end{aligned}$$

Do you see something here? Do you see the P_0? Here we go. Squares are positive:

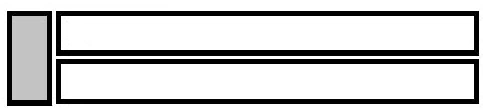
$$\begin{aligned}(m-n)^2&\geq 0\\ \Longrightarrow m^2-2mn+n^2&\geq 0\\ \Longrightarrow 2m^2+2n^2&\geq m^2+2mn+n^2\\ \Longrightarrow 2(m^2+n^2)&\geq(m+n)^2\\ \Longrightarrow \frac{m^2+n^2}{m+n}&\geq\frac{m+n}{2}\end{aligned}$$

The only time you get an equality is if $m=n$.

10. Consider an A4 (210 mm × 297 mm) cardboard sheet. Suppose the cardboard is 1 mm thick. Prove that the sheet cannot be folded nine times.

Solution: A reasonable assumption is that when we fold the paper, there is a folding zone:

Approximately, the length of this folding zone must match the length of two thicknesses:



Now each fold doubles the thickness of the paper, $t_{n+1} = 2t_n$, and thus doubles the length of the folding zone, but also reduces the height/width by twice the previous thickness, before we fold it. However the previous thickness is also added to the length.

$$l_{n+1} = \frac{l_n - 2t_n}{2} + t_n = \frac{1}{2} l_n.$$

Assuming we fold the long side first, we can write down a *recurrence* for what happens to the height h_n, width w_n, and thickness t_n after n folds:

$$\begin{pmatrix} h_{n+1} \\ w_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} 297 \\ 210 \\ 1 \end{pmatrix}, & \text{if } n = 0 \\ \begin{pmatrix} \frac{1}{2} h_n \\ w_n \\ 2t_n \end{pmatrix}, & \text{if } n \text{ odd} \\ \begin{pmatrix} h_n \\ \frac{1}{2} w_n \\ 2t_n \end{pmatrix}, & n \text{ even.} \end{cases}$$

The problem is that the folding zone becomes longer than half the height/width. In fact after the fifth fold this happens:

fold, n	height, h_n/mm	width, w_n/mm	thickness t_n/mm	length of folding zone/mm
0	297	210	1	2
1	148.5	210	2	4
2	148.5	105	4	8
3	74.25	105	8	16
4	74.25	52.5	16	32
5	37.125	$\underline{52.5}$	32	$\underline{64}$
6	-	-	64	128
7	-	-	128	256
8	-	-	256	512
9	-	-	512	1024

Also, after eight folds, the folding zone is longer than either of the original height or width.